

# Magnetic properties of the Larkin-Ovchinnikov-Fulde-Ferrell superconducting phase

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We compute, at the first order in the fine structure constant, the parameters of the electromagnetic Lagrangian for the inhomogeneous Larkin-Ovchinnikov-Fulde-Ferrell phase in Quantum Chromodynamics (QCD) and in condensed matter. In particular we compute for QCD with two flavors the dielectric and the magnetic permeability tensors, and for condensed matter superconductors the penetration depth of external magnetic fields.

## I. INTRODUCTION

The aim of this paper is to compute, at the first order in the fine structure constant, the parameters of the electromagnetic Lagrangian for the inhomogeneous Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) [1, 2] phase in QCD and in condensed matter. In particular we compute in 2 Flavor QCD the dielectric constant and the magnetic permeability. For the condensed matter superconductor we compute the penetration depth of an external magnetic field at  $T = 0$ . The inhomogeneous superconductive LOFF phase was introduced in the context of ordinary superconductors forty years ago. In the original papers [1, 2] this phase

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was discussed for weak ferromagnetic materials with an exchange interactions produced by the presence of paramagnetic impurities. It was shown that the spin splitting generates a separation of the Fermi surfaces. This separation, here denoted as  $\delta\mu$ , is proportional in metallic superconductors to the magnetic field  $H$ . For large enough  $\delta\mu$ , beyond the so called Clogston-Chandrasekhar limit  $\sim \Delta_0/\sqrt{2}$  ( $\Delta_0$  the gap for the homogeneous BCS case) [3, 4], it can be energetically favorable for two electrons to form a pair with non vanishing total momentum  $|2\mathbf{q}|$ . The main effect is an inhomogeneous gap with a space modulation which depends on its plane wave decomposition. The LOFF phase can exist also in QCD as a particular realization of color superconductivity at non asymptotic densities, due to difference in Fermi momenta, as arising from different quark masses and from  $\beta$  equilibrium in dense quark matter. Translation and rotation invariance are broken and the space dependence of the order parameter may be that of a crystal. Such a crystalline phase of QCD might occur in compact stars, and suggestions exist that it may explain the variation patterns in the pulsars rotation periods (glitches) [5].

The range of densities where the LOFF phase might be energetically favored is still matter of debate. In a recent paper [6] the intermediate density region has been studied and the possible spatially uniform candidate phases have been examined. The conclusion is in favor of a gapless CFL phase (gCFL) and, for immediately lower densities, in favor of the LOFF phase, based on the indications of the calculation in Ref. [7]. These results, if confirmed, would make more likely the occurrence of the LOFF phase at the pre-asymptotic densities of compact stars.

An important point is the form of the condensate. Recent analyses [7, 8] point to cubic structures as the energetically favored form of the condensate. They are the body-centered cube (bcc) and the face-centered cube (fcc), obtained summing 6 or 8 plane waves pointing to the faces or the vertices of a cube. The bcc structure seems the dominant one for  $\delta\mu$  near the Clogston limit [7]; for larger values of  $\delta\mu$  the fcc structure is favored, [7, 8]. Therefore here we consider three different structures, i.e. the one plane wave case (Fulde-Ferrell phase) and the two cubic structures. We perform our study in a well defined approximation, not based on the Ginzburg-Landau approach, but valid for  $\Delta$  not too small [7]. This approximation is based on a convenient average over the sites of the crystalline structure defined by the condensate. The result can be described in terms of a multi-valued gap function possessing  $P$  branches, where  $P$  is the number of plane waves defining the crystal. Each of these

branches corresponds to a gap  $k\Delta$ ,  $k = 1, \dots, P$  with  $\Delta$  the constant gap factor appearing in the LOFF condensate.

In Sections II and III we review the formalism employed to describe the LOFF phase and the results obtained by the approximation of ref. [7]. In Section IV we discuss the LOFF phase in QCD. We discuss the problem of the Meissner mass and the determination of the parameters (dielectric constant and magnetic permeability) of the Lagrangian for the electromagnetic field. Differently from the homogeneous two flavor case (2SC) [9], we find a correction not only for the dielectric constant, but also for the magnetic permeability. Since in QCD there is a rotated  $U(1)$  that is conserved [10], this implies a constraint on our calculation scheme since the Meissner mass must vanish. We use this result in Section IV where we consider the LOFF phase in condensed matter. In this case the Meissner mass does not vanish and in general the magnetic field  $H$  should be expelled. This has been discussed in the LOFF superconductor [1] within the Ginzburg-Landau approximation for the case of a gap with a space modulation  $\sim \Delta \cos 2qz$  and in [2] for the one-plane-wave case. In this Section we consider other crystalline structures and the region near the Clogston limit, far away from the second order transition point.

## II. GENERAL FORMALISM

In this Section we briefly review the formalism we employ to describe inhomogeneous color superconductivity in QCD; modifications for condensed matter applications will be discussed in Section IV. We consider QCD with two massless quarks having different chemical potentials  $\mu_1$  and  $\mu_2$  and we suppose that  $\delta\mu = |\mu_1 - \mu_2|/2$  is slightly larger than the Clogston-Chandrasekhar [3, 4] limit  $\Delta_0/\sqrt{2}$ , where  $\Delta_0$  is the value of the gap for the homogeneous BCS phase. We work in zero temperature high quark density limit, which means that  $\mu = (\mu_1 + \mu_2)/2 \gg \delta\mu$ . In these hypotheses the system can be supposed to be in the LOFF phase characterized by the following pattern of condensation:

$$\langle \psi_{i\alpha} C \psi_{j\beta} \rangle = \Delta \sum_{m=1}^P e^{2iq\mathbf{n}_m \cdot \mathbf{r}} \epsilon^{\alpha\beta 3} \epsilon_{ij} \quad (1)$$

where  $\alpha, \beta$  are color indices,  $i, j$  are flavor indices and  $2q\mathbf{n}_m$  is the total momentum of the Cooper pair. We will consider below three cases. The first is the one-plane wave Fulde Ferrel state with  $P = 1$  (and  $\mathbf{n}$  along the  $z$ -axis). In the second case we take  $P = 6$  with the

six unit vectors  $\mathbf{n}_m$  pointing to the six faces of a cube (bcc). Finally we consider the case  $P = 8$  with the eight unit vectors  $\mathbf{n}_m$  pointing to the eight vertices of a cube (fcc). With the choice of phases for the plane waves as in Eq. (1) the symmetry of the condensate both for  $P = 6$  and  $P = 8$  corresponds to the cube group.

The reason to discuss only these cases is based on the results of [7]. Here it is shown that the bcc is the energetically favored structure in the  $\delta\mu$  interval  $(0.707\Delta_0 - 0.95\Delta_0)$ , while for  $\delta\mu$  in the interval  $(0.95\Delta_0 - 1.12\Delta_0)$  the fcc dominates. The approximation used in [7] is based on the so called High Density Effective Theory (HDET) [11, 12, 13, 14, 15, 16] and on an averaging procedure of the original Lagrangian over a region of the size of the lattice cell. In the HDET formulation one decomposes the fermion momentum  $p^\mu$  in its hard part  $\mu v^\mu$  and a residual momentum  $\ell^\mu$ , i.e.  $p = \mu v + \ell$  where  $v^\mu = (0, \mathbf{v})$ ;  $\mathbf{v}$  is the fermion velocity and one neglects in  $\ell$  the transverse part writing  $\ell = (\ell_0, \ell_\parallel \mathbf{v})$ , with  $\ell_\parallel = \boldsymbol{\ell} \cdot \mathbf{v}$ . Since  $\mu_1 \neq \mu_2$  we have two velocities here, but one can prove that in the large  $\mu$  limit,  $\mathbf{v}_1 = -\mathbf{v}_2 + \mathcal{O}(\delta\mu/\mu)$ . The momentum decomposition allows to define velocity-dependent fields, whose Fourier transform depends on  $\ell$ . The averaging procedure substitutes the inhomogeneous gap  $\Delta(\mathbf{r}) \propto \sum \exp(2iq\mathbf{n}_m \cdot \mathbf{r})$  with a function of  $\ell$  and  $\mathbf{v}$ ; the whole approach is justified if the velocity dependent fields are slowly varying over regions of the order of the lattice size. Therefore this Lagrangian can only describe soft momenta. For more details see [7].

Let us write the fermion propagator in this approach. Since we have four degrees of freedom (two flavors and two colors) we can use a compact notation introducing a base of velocity-dependent fermion fields  $\psi_A$  with  $A = 1, \dots, 4$ . In this base the quark propagator assumes the form

$$S_{AB}(\mathbf{v}, \ell) = \frac{1}{D(\mathbf{v}, \ell)} \begin{pmatrix} \tilde{V} \cdot \ell \delta^{AB} & -\Delta_{AB} \\ -\Delta_{AB} & V \cdot \ell \delta^{AB} \end{pmatrix} \quad (2)$$

with  $A, B = 1, \dots, 4$ . Here  $V = (1, \mathbf{v})$ ,  $\tilde{V} = (1, -\mathbf{v})$ , the gap matrix is

$$\Delta_{AB} = \Delta_E(\mathbf{v}, \ell_0) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

and

$$D(\mathbf{v}, \ell) = V \cdot \ell \tilde{V} \cdot \ell - \Delta_E^2(\mathbf{v}, \ell_0). \quad (4)$$

The effective gap is given by

$$\Delta_E(\mathbf{v}, \ell_0) = \sum_{m=1}^P \Delta_{eff}(\mathbf{v} \cdot \mathbf{n}_m, \ell_0) , \quad (5)$$

with

$$\Delta_{eff}(\mathbf{v} \cdot \mathbf{n}, \epsilon) = \Delta\theta(E_u)\theta(E_d) . \quad (6)$$

Here

$$E_{u,d} = \pm\delta\mu \mp q\mathbf{v} \cdot \mathbf{n} + \epsilon \quad (7)$$

are the quasi-particle dispersion laws and  $\epsilon$  is the value of the energy at the pole of the propagator.

### III. $\tilde{U}(1)$ GAUGE INVARIANCE AND THE LOFF PHASE OF QCD

The quark pair condensate (1) breaks the electromagnetic group  $U(1)_{em}$  since the pairs have total non-zero electric charge. There exists however a group which we call  $\tilde{U}(1)$  generated by a linear combination of the electromagnetic charge  $Q$  and the  $T_8(= \lambda_8/2)$  color generator

$$\tilde{Q} \equiv \tilde{Q}_{ij}^{\alpha\beta} = Q_{ij} \otimes I^{\alpha\beta} - \frac{1}{\sqrt{3}} I_{ij} \otimes T_8^{\alpha\beta} \quad (8)$$

that remains unbroken as far as the transverse degrees of freedom are concerned. The residual symmetry embodied by Eq.(8) implies a mixing angle  $\theta$  between the photon  $\mathbf{A}$  and gluon  $\mathbf{G}^8$ :

$$\cos \theta = \frac{g \sqrt{3}}{\sqrt{3}g^2 + e^2} ; \quad (9)$$

The *in-medium* vector potential fields  $\tilde{\mathbf{A}}$  and  $\tilde{\mathbf{G}}^8$  are then given by

$$\tilde{A}_i = -\sin \theta G_i^8 + \cos \theta A_i , \quad (10)$$

$$\tilde{G}_i^8 = \cos \theta G_i^8 + \sin \theta A_i . \quad (11)$$

This phenomenon is similar to what happens in the two-flavor superconducting phase of QCD, the so-called 2SC model, where the quark pair condensate has the same color and flavor dependence of (1), but with  $q = 0$ . These results indeed do not depend on the space modulation of the condensate. Gauge invariance under  $\tilde{U}(1)$  implies that the polarization tensor  $\Pi_{ij}(p)$  of the  $\tilde{A}_i$  field vanishes for zero external momentum  $p = 0$ . To check this result

at the one loop level in the HDET we consider the self-energy and tadpole diagrams (see e.g. Fig.1 in [14]) . They are computed by the propagator (2) and the effective interaction Lagrangians

$$\mathcal{L}_{qq\tilde{A}} = i\tilde{e} \sum_{\mathbf{v}} \chi_A^\dagger \begin{pmatrix} -\mathbf{v} \cdot \tilde{\mathbf{A}} \tilde{Q}_{AB} & 0 \\ 0 & -\mathbf{v} \cdot \tilde{\mathbf{A}} \tilde{Q}_{AB} \end{pmatrix} \chi_B , \quad (12)$$

$$\mathcal{L}_{qq\tilde{A}\tilde{A}} = -\tilde{e}^2 \sum_{\mathbf{v}} \chi_A^\dagger \begin{pmatrix} \frac{\tilde{Q}_{AB}^2}{2\mu + \tilde{V} \cdot \ell} & 0 \\ 0 & \frac{\tilde{Q}_{AB}^2}{2\mu + \tilde{V} \cdot \ell} \end{pmatrix} \chi_B P^{ij} \tilde{A}_i \tilde{A}_j \quad (13)$$

where  $\tilde{e} = e \cos \theta$ ,

$$P_{\mu\nu} = g_{\mu\nu} - \frac{1}{2} (V_\mu V_\nu + \tilde{V}_\mu \tilde{V}_\nu) , \quad (14)$$

and  $\tilde{Q} = 1/2 \text{diag}(+1, +1, -1, -1)$ .

For any crystalline structure we have the result

$$\begin{aligned} i \Pi_{ij}(p) &= \frac{\tilde{e}^2 \mu^2}{12 \pi^3} \int \frac{d\mathbf{v}}{4\pi} \int d^2\ell v_i v_j \frac{V \cdot \ell V \cdot (\ell + p) + \tilde{V} \cdot \ell \tilde{V} \cdot (\ell + p) - 2 \Delta_E^2}{D(\ell) D(\ell + p)} \\ &- \frac{\tilde{e}^2}{6 \pi^3} \int \frac{d\mathbf{v}}{4\pi} \int d^2\ell P_{ij} \left[ (\mu + \ell_\parallel)^2 \frac{\tilde{V} \cdot \ell}{(2\mu + \tilde{V} \cdot \ell) D(\ell)} + (V \rightarrow \tilde{V}) \right] . \end{aligned} \quad (15)$$

The former term on the r.h.s. is the contribution of the self-energy graph, the latter term is from the tadpole diagram.

For the Fulde Ferrel one plane wave case we can distinguish two contributions, one from the pairing region, and the other from the blocking region. The pairing region is defined by ( $\xi = \ell_\parallel$ ):

$$\mathcal{P}_1 = \left\{ (\xi, \mathbf{v}) \mid \Delta = \Delta_{eff}(\mathbf{v} \cdot \mathbf{n}, \epsilon) \big|_{\epsilon = \sqrt{\xi^2 + \Delta^2}} \right\} . \quad (16)$$

The blocking region  $\mathcal{P}_0$  corresponds to  $E_u < 0$  or  $E_d < 0$  or both. We have heuristically tested the vanishing of the Meissner mass in this case by computing numerically  $\tilde{\Pi}_{ij}(0)$  for the following values of the parameters  $\mu = 400$  MeV,  $\delta\mu = \delta\mu_1 = \Delta_0/\sqrt{2}$  and taking the values of  $\delta\mu/q = 0.78$  and of  $\Delta = 0.24\Delta_0$  that minimize the free energy [7]. We find  $\tilde{\Pi}(0)$  of the order of  $10^{-3}$ , while the two individual contributions are of the order of 1, which points to an almost complete cancellation. This result was expected on the basis of gauge invariance and the rather accurate approximation employed to get (7), i.e.  $\delta\mu/\mu \ll 1$ .

Next we consider the case of several plane waves with wave vectors with the same modulus  $q$ , but directed along the directions  $\mathbf{n}_m$ ,  $m = 1, \dots, P$ . Here our formalism is based on the

approach discussed in Section II. We have different regions  $\mathcal{P}_k$  where pairing is possible. They are defined as follows

$$\mathcal{P}_k = \{(\mathbf{v}, \xi) \mid \Delta_E(\mathbf{v}, \epsilon) = k\Delta\}, \quad k = 1, \dots, P \quad (17)$$

where  $\Delta_E$  is in (5) and one uses (7) with  $\mathbf{n} \rightarrow \mathbf{n}_m$ . This approximation is the result of an averaging procedure described in detail in [7] and is valid for  $\xi$  of the order of  $q$  or smaller. Now in the self energy term in the r.h.s of (15) the relevant contribution in the  $\xi$  integration comes from the small  $\xi$  region; therefore the approximation is adequate. On the other hand in the second term (tadpole contribution) the hard modes dominate and the approximation is no longer valid. Therefore we use  $\tilde{U}(1)$  gauge invariance to get information on the main features of the pairing regions for large  $\xi$ . Notice that the sum over  $k$  arises because plugging (5) into (15) one has several terms, corresponding to different values assumed by the gap:  $k\Delta$  ( $k = 1, \dots, P$ ). Imposing gauge invariance one gets

$$0 = \sum_k \left\{ \frac{1}{2} \int_{\mathcal{P}_k} \frac{d\xi d\mathbf{v}}{4\pi} \frac{k^2 \Delta^2}{(\xi^2 + k^2 \Delta^2)^{3/2}} + \int_{\tilde{\mathcal{P}}_k} \frac{d\xi d\mathbf{v}}{4\pi \sqrt{\xi^2 + k^2 \Delta^2}} \left( \frac{(1 - \xi/\mu)^2 (\sqrt{\xi^2 + k^2 \Delta^2} + \xi)}{\sqrt{\xi^2 + k^2 \Delta^2} + \xi - 2\mu} + (\xi \leftrightarrow -\xi) \right) \right\} \quad (18)$$

where  $\tilde{\mathcal{P}}_k$  is the region of the phase space where  $\Delta_E = k\Delta$ , but  $\xi \sim \mu$ . Even if this equation does not determine  $\tilde{\mathcal{P}}_k$ , it provides sufficient information to compute the Meissner mass in ordinary LOFF superconductors. An application of this result is discussed in the next Section.

For small external momentum we get from Eq.(15) and from the condition of vanishing Meissner mass, that

$$i\Pi(p)_{ij} \approx -\frac{\tilde{e}^2 \mu^2}{12\pi^3} [A_{ij} p_0^2 + B_{ij}^{kl} p_k p_l] \quad (19)$$

where we have defined

$$A_{ij} = \sum_{k=1}^P \int_{\mathcal{P}_k} \frac{d^2 \ell d\mathbf{v}}{4\pi} v_i v_j \frac{2}{D^2(l)}, \quad B_{ij}^{kl} = \sum_{k=1}^P \int_{\mathcal{P}_k} \frac{d^2 \ell d\mathbf{v}}{4\pi} v_i v_j v_k v_l \left[ \frac{2}{D^2(l)} + \frac{4\Delta_E^2}{D^3(l)} \right]. \quad (20)$$

For the FF state we have two independent transverse tensors,  $\Pi_1^T = \Pi_{11}$  and  $\Pi_3^T = \Pi_{33}$ , while for both cubic structures we have

$$\Pi^T(p) = \frac{1}{2} \left( \delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2} \right) \Pi_{ij}(p) \quad (21)$$

as in the homogeneous case. The Lagrangian for the rotated photon can be written in the form

$$\mathcal{L} = \frac{1}{2} \left( \epsilon_{ij} E_i E_j - \frac{1}{\lambda_{ij}} B_i B_j \right), \quad (22)$$

where  $\epsilon_{ij}$  and  $\lambda_{ij}$  can be obtained from Equations (20). In general we have

$$\epsilon_{ij} = \left( 1 + f_j(\delta\mu, \Delta_0) \frac{\tilde{e}^2 \mu^2}{18\pi^2 \Delta^2} \right) \delta_{ij}, \quad \lambda_{ij}^{-1} = \left( 1 + g_j(\delta\mu, \Delta_0) \frac{\tilde{e}^2 \mu^2}{18\pi^2 \Delta^2} \right) \delta_{ij}. \quad (23)$$

The coefficients  $f_j, g_j$  assume different values according to the crystalline structure. At  $\delta\mu = \delta\mu_1$  and  $\Delta_0 = 40$  MeV we have, for the one-plane-wave (FF)  $f_1 = f_2 = +0.12$ ,  $f_3 = +0.23$ ,  $g_1 = g_2 = +0.31 \times 10^{-2}$  and  $g_3 = +0.13 \times 10^{-3}$ . For the body-centered-cube  $f_{bcc} = +0.49$  and  $g_{bcc} = -0.09$ . For the face-centered-cube at  $\delta\mu = 0.95\Delta_0$  we have  $f_{fcc} = +0.46$  and  $g_{fcc} = -0.09$ . For the 2SC case we have, in agreement with [9, 17],  $f_{2SC} = 1$  and  $g_{2SC} = 0$ , showing absence of corrections for the magnetic permeability in the homogeneous phase.

#### IV. PENETRATION DEPTH IN CONDENSED MATTER

As an application of the result (18), in this Section we give an estimate of the penetration depth of a weak static magnetic field in an ordinary condensed-matter LOFF superconductor at  $T = 0$ . We assume that the field is small enough to produce an exchange field and a sizeable paramagnetic effect so that the Fermi surfaces of the two pairing electrons are separated and the LOFF phase is formed (in particular the Clogston-Chandrasekhar limit is reached). At the same time we assume that the effect of the external field can be neglected in the gap equation. This is a strong assumption, as we know that the effect of an external magnetic field is a modulation of the gap, see e.g. [18]. As a matter of fact the paramagnetic effect, which is needed to produce the separation of the Fermi surfaces, and the diamagnetic effect, which is detrimental to superconductivity, are in general related. Therefore the original proposal of [1, 2] is now considered only as an ideal case. The actual experimental activity points to layered superconductors where one can minimize diamagnetic effects by choosing the external magnetic field parallel to the layer. We refer the interested reader to the specialized literature (see e.g. [19] and references therein) for a discussion. For the time being we study the effect of the magnetic field in the idealized case where its effects on the gap equations can be neglected.

Let us assume that a plane surface ( $yz$  plane) divides the space into two parts, one containing the superconductor in the LOFF phase (half-space with  $x > 0$ ) and the other containing matter in the normal phase. At the interface the magnetic field  $\mathbf{H}$  is parallel to the  $yz$  plane. We take  $\mathbf{H}$  along the  $z$ -axis and the vector potential  $\mathbf{A}$  directed along the  $y$ -axis;  $\mathbf{A}$  depends only on  $x$  and we assume  $\nabla \cdot \mathbf{A} = 0$ .

In condensed matter the LOFF condensate has the form similar to (1):

$$\langle \psi C \psi \rangle = \Delta(\mathbf{r}) = \Delta \sum_{m=1}^P e^{2i q \mathbf{n}_m \cdot \mathbf{r}} \quad (24)$$

where the  $\psi$  are non relativistic, two components spinor fields describing electrons. The Lagrangian can be written as follows [19]

$$\mathcal{L} = \sum_{\mathbf{v}} \chi_a^\dagger \begin{pmatrix} (V \cdot \ell + e \mathbf{v} \cdot \mathbf{A}) \delta^{ab} + \delta\mu \sigma_3^{ab} & -\Delta_E(\mathbf{v}, \ell_0) \delta^{ab} \\ -\Delta_E(\mathbf{v}, \ell_0) \delta^{ab} & (\tilde{V} \cdot \ell - e \mathbf{v} \cdot \mathbf{A}) \delta^{ab} + \delta\mu \sigma_3^{ab} \end{pmatrix} \chi_b, \quad (25)$$

where we have used the effective  $\Delta_E(\mathbf{v}, \ell_0)$  approximation;  $\delta\mu$  is proportional to the exchange field acting on the electron spin and the term  $\delta\mu \sigma_3$  describes a paramagnetic coupling. In (25)  $a, b$  are spin indices and  $\chi_a$  are Nambu-Gorkov fields. The Lagrangian includes the coupling to the external vector potential field  $\mathbf{A}$ . The penetration depth is defined by [20]

$$\delta = \frac{1}{H_0} \int_0^\infty dx H(x), \quad (26)$$

where  $H_0$  is the value of the magnetic field outside the superconductor. If rotational symmetry holds (the BCS and the cubic structures) one gets, using previous hypotheses [20]:

$$\delta = \frac{2}{\pi} \int_0^\infty dp \frac{1}{p^2 - \Pi_T(p)} \quad (27)$$

where  $\Pi_T(p)$  is computed by (21) in the static  $p_0 = 0$  approximation; for  $\Pi_{ij}(\mathbf{p})$  we have

$$\Pi_{ij}(\mathbf{p}) = -\frac{e^2 p_F m v^2}{4 \pi^2} \sum_k \int_{-\infty}^{+\infty} d\xi \int_{\mathcal{P}_k} \frac{d\hat{\mathbf{v}}}{4 \pi} \hat{v}_i \hat{v}_j J(\xi, \beta, k \Delta) + \delta \Pi_{ij}, \quad (28)$$

where we have defined

$$J(\xi, \beta, k \Delta) = \frac{1}{\xi \beta} \left[ \frac{\xi^2 - \beta \xi + k^2 \Delta^2}{\sqrt{(\xi - \beta)^2 + k^2 \Delta^2}} - \frac{\xi^2 + \beta \xi + k^2 \Delta^2}{\sqrt{(\xi + \beta)^2 + k^2 \Delta^2}} \right], \quad (29)$$

with  $\beta = \mathbf{p} \cdot \hat{\mathbf{v}}/2$  and  $\hat{\mathbf{v}}$  is the direction of Fermi velocity  $\mathbf{v}$ . The sum over  $k$  in (28) goes from  $k = 0$  to  $k = P$ . For the BCS homogeneous case there is one term:  $k = 1$ . For the

one-plane wave case there are two terms, one with  $k = 0$  corresponding to the blocking region, where  $\Delta = 0$ , and the other one with  $k = 1$ , corresponding to the pairing region. For all the other cases the sum runs from  $k = 1$  to  $k = P$ . As a matter of fact, within our approximation [7], for the structures with more than one plane wave, pairing is possible in the whole phase space. In these cases, as discussed in Section IV.A of [7], one can identify the blocking region with the domain where only the branch with gap  $P\Delta$  of the dispersion law contributes. The two terms correspond to the two terms on the r.h.s. of Eq. (15).

We are interested in Type II superconductors where the relevant momenta in Eq. (27) are  $p \simeq 0$ . For the BCS case one gets

$$\delta \approx \frac{2}{\pi} \int_0^\infty dp \frac{1}{p^2 + m_M^2} = \frac{1}{m_M} \quad (30)$$

where  $m_M^2 = \frac{e^2 p_F^2}{3\pi^2} v$  is the squared Meissner mass in the BCS phase.

Let us consider now the LOFF phase. For the one-plane-wave case we have

$$\Pi_{ij}(\mathbf{p}) = -m_M^2 \left( \delta_{ij} - 3 \int_{\mathcal{P}_0} \frac{d\hat{\mathbf{v}}}{4\pi} \hat{v}_i \hat{v}_j + \frac{3}{4} \int d\xi \int_{\mathcal{P}_1} \frac{d\hat{\mathbf{v}}}{4\pi} \hat{v}_i \hat{v}_j J\left(\xi, \frac{\mathbf{p} \cdot \hat{\mathbf{v}}}{2}, \Delta\right) \right). \quad (31)$$

The dependence on the total momentum of the Cooper pair  $2\mathbf{q}$  is in the definition of  $\mathcal{P}_1$ . It is convenient to consider the tensor  $\tilde{\Pi}$  with  $\mathbf{q}$  along the  $x$ -axis. The relation between the two tensors is (sum over  $k, l$ )

$$\Pi_{ij}(\mathbf{p}) = R_{ik}(\theta) R_{jl}(\theta) \tilde{\Pi}_{kl}(\mathbf{p}), \quad (32)$$

where  $R_{ij}(\theta)$  is the rotation matrix which brings  $\mathbf{q}$  along the  $x$  axis, that is along  $\mathbf{p}$  (the direction of the gradient of the magnetic field).  $\tilde{\Pi}$  has two independent components  $\tilde{\Pi}_{11} = \tilde{\Pi}_{22}$  and  $\tilde{\Pi}_{33}$  and the superconductor is characterized by two independent penetration lengths which we compute in the London limit. We do that for the following values of the parameters [7]:  $\delta\mu = \delta\mu_1$ ,  $z_q = 0.78$  and  $\Delta = 0.24\Delta_0$ . In this case we get:  $\delta_1 \simeq 2.6\delta_L$  and  $\delta_3 \simeq 1.4\delta_L$  where  $\delta_L$  is the London penetration depth in the BCS case. Within our approximation, consisting in neglecting terms of the order  $\delta\mu/\mu$  our results are compatible with those of [2]; for example for  $\delta_3/\delta_L$  we find agreement with [2] within 10%.

We notice that for  $\Delta \rightarrow 0$ , near the second order phase transition, the pairing region vanishes, whereas the blocking region  $\mathcal{P}_0$  is the whole Fermi surface. From (31) we see that  $\Pi_{ij}(0)$  vanishes, and from (27) we get that both  $\delta_i$  diverge, which means that the FF is no longer a superconductor, in agreement with the result of [1].

We can repeat the analysis for cubic crystalline structures (bcc and fcc). In these cases one can exploit the residual discrete symmetry and only one penetration length is present.

The transverse component of the polarization tensor is obtained by (28) using the appropriate set of plane waves. In the London limit only  $p \sim 0$  are relevant and one gets:

$$\begin{aligned}\Pi_T(0) &= 2m_M^2 \sum_{k=1}^P \int_{\tilde{\mathcal{P}}_k} \frac{d\xi d\hat{\mathbf{v}}}{4\pi\sqrt{\xi^2 + k^2\Delta^2}} \left( \frac{(1 - \xi/\mu)^2(\sqrt{\xi^2 + k^2\Delta^2} + \xi)}{\sqrt{\xi^2 + k^2\Delta^2} + \xi - 2\mu} + (\xi \leftrightarrow -\xi) \right) \\ &= -m_M^2 \sum_{k=1}^P \int_{\mathcal{P}_k} \frac{d\xi d\hat{\mathbf{v}}}{4\pi} \frac{k^2\Delta^2}{(\xi^2 + k^2\Delta^2)^{3/2}}.\end{aligned}\quad (33)$$

In the second line we have used the result expressed by Eq. (18). Numerically we get for the London penetration length in the bcc case  $\delta \approx 0.69\delta_L$  at  $\delta\mu = \delta\mu_1$ ; for the fcc case we get  $\delta \approx 0.52\delta_L$  at  $\delta\mu = 0.95\Delta_0$ , where, according to [7] there is a transition from the bcc to the fcc LOFF phase.

## V. CONCLUSIONS

We have used the high density effective theory formalism to compute the low energy properties of the electromagnetic Lagrangian of the LOFF phase in QCD and condensed matter. We have shown that in QCD the rotated photon associated to the unbroken  $\tilde{U}(1)$  group is screened both electrically and magnetically. We have computed near the Chandrasekhar-Clogston point the dielectric tensor and the magnetic permeability tensor for the one-plane-wave, the body-centered cube and the face-centered cube crystalline structures. In condensed matter we have computed the penetration depth of an external magnetic field. In the London limit the penetration depth is proportional to the London penetration depth of the BCS case with coefficients that assume different values according to the crystalline structure.

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